Lecture 3 2012/10/15

Outline

1. **Motivation**
2. **Picard Theorem**
3. **Peano Theorem**
4. **Summary**
5. **Appendices**

1. Motivation

In an experiment on a physical system we expect:

- **(Existence)** that stating from an initial state $x(t_0)$, the state will move and $x(t)$ will be defined in (at least immediate) future time $t > t_0$;
- **(Uniqueness)** to have exactly the same behavior if we repeat the experiment in the same way.

The mathematical model of such a physical system:

$$\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases} \qquad (E)$$

needs to exhibit these two properties: existence and uniqueness of solutions!

Remark 3.1 If we can solve (E), then it is simple to study these two properties and the dynamical behavior of (E) because everything is known. However, in most cases, we can't solve it in an explicit form. Let us see the example 3.1.

Example 3.1 Consider the Riccati equation as follows.

$$\begin{cases} x' = x^2 + t^2 \\ x(0) = 0 \end{cases}.$$

it can't be solved in an explicit way, however, there exists a unique solution by a geometric way shown by Fig. 3.1.

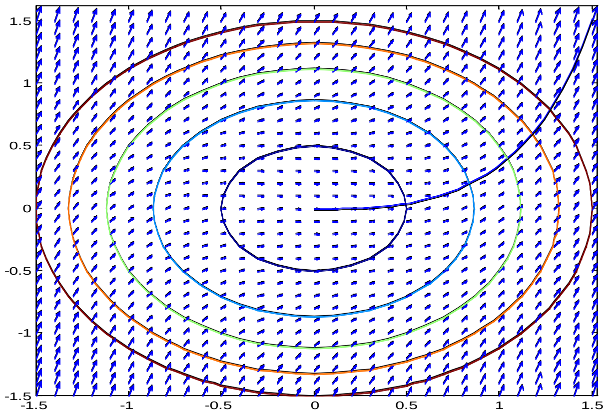

Fig. 3.1

Remark 3.2 Example 3.1 shows that we have to change the research direction of ODE and develop some effective methods to explore solution properties and the other dynamical behaviors based on the known information f without solving differential equations.

Example 3.2 Let us see the IVP:

$$\begin{cases} x' = \delta(x) \\ x(0) = 0 \end{cases}, \quad \delta(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

When $x \neq 0$, $\delta(x) \equiv 0$, i.e. $x' = 0 \Rightarrow x(t) = c \neq 0$. It is shown that the initial condition $x(0) = 0$ can't be satisfied. When $x = 0$, $\delta(x) = 1$, i.e. $x' = 1 \Rightarrow$ this contradicts $x = 0$. Therefore, there is no solution of the IVP.

Remark 3.3 Example 3.2 shows that for sure of existence, we need the continuity of f. In fact, it guarantees existence only! – **Peano Theorem**.

Example 3.3 The IVP

$$\begin{cases} x' = \sqrt{x} \\ x(0) = 0 \end{cases}$$

has two different solutions: $x(t) \equiv 0$ and $x(t) = \dfrac{t^2}{4}$.

Remark 3.4 Example 3.3 shows that the continuity of f is not enough for uniqueness of solution. In general, we need a Lipschitz condition. – **Picard Theorem**.

In this chapter, we basically study the local properties of (E). So we study (E) on the compact set $Q=\{(t,x)\in R\times R^n: |t-t_0|\le a,\ \|x-x_0\|\le b\}$.

2. Picard Theorem

1) Lipschitz Condition

We say that $f: Q\to R^n$ satisfies a Lipschitz condition on Q if

$$\|f(t,x)-f(t,y)\|\le L\|x-y\|, \text{ for any } (t,x),(t,y)\in Q.$$

Remark 3.5 It is not easy in general to verify the Lipschitz condition by definition. However, if $\frac{\partial f}{\partial x}(t,x)$ is continuous on Q, then we can take

$$L\ge \max_{(t,x)\in Q}\|\frac{\partial f}{\partial x}(t,x)\|,$$

where $\frac{\partial f}{\partial x}=\left(\frac{\partial f_j}{\partial x_i}\right)_{i,j=1,n}$ is the Jacobian matrix of f. Therefore,

$\frac{\partial f}{\partial x}$ is continuous on Q $\Rightarrow$ f satisfies a Lipschitz condition on Q.

However, the opposite is not true! e.x. $f(t,x)=|x|$ at $x=0$.

2) The statement of Picard Theorem (Existence and Uniqueness)

Picard Theorem. Suppose that $f: Q\to R^n$

- is continuous;
- satisfies a Lipschitz condition.

Then (E) has a unique solution $x(t)$ at least defined on $\bar{I}=[t_0-\bar{h},t_0+\bar{h}]$, where

$\bar{h}=\min\{h,\frac{1}{L}\}$, $h=\min\{a,\frac{b}{M}\}$, $M=\max_{(t,x)\in Q}\|f(t,x)\|$.

Proof. (Using Banach Fixed Point Theorem)

Step 1. Define a Banach space $C(\bar{I})$, where $\bar{I}=[t_0-\bar{h},t_0+\bar{h}]$, with a norm given by $\|x\|_\infty=\max_{t\in\bar{I}}\|x(t)\|$, $\forall x\in C(\bar{I})$.

Step 2. Define a subset of $C(\bar{I})$ as follows

$$D = \{x : x \in C(\overline{I}),\ \| x(t) - x_0 \|_\infty \le b,\ t \in \overline{I}\}.$$

Show $D \subset C(\overline{I})$ is closed. For any $x_n \in D$ with $\| x_n - x \|_\infty \to 0\ (n \to \infty)$, since $\| \cdot \| : R^n \to R$ is a continuous mapping, $\lim_{n\to\infty} \| x_n(t) - x_0 \|_\infty \le b$ yields $\| x(t) - x_0 \|_\infty \le b$. i.e. $x \in D$. Therefore, D is closed.

Step 3. Define a mapping $T : D \to D$ as follows.

$$(Tx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds,\quad \forall t \in \overline{I},\quad \forall x \in D.$$

Step 4. Verify T satisfies two conditions of Banach fixed point theorem: $T(D) \subseteq D$ and the contractive condition.

- Show $T(D) \subseteq D$. Since

$$\| (Tx)(t) - x_0 \| \le \int_{t_0}^{t} \| f(s, x(s)) \| ds \le M\, | t - t_0 | \le M\overline{h} \le Mh \le b,\ \forall t \in \overline{I},\ \forall x \in D,$$

and $\overline{I}$ is compact, then $\max_{t\in\overline{I}} \| (Tx)(t) - x_0 \| \le b$. i.e. $\| Tx - x_0 \|_\infty \le b$, which implies $Tx \in D$. Therefore, $T(D) \subseteq D$.

- Show the contractive condition. For any $x_1, x_2 \in D$, we have (with the Lipschitz condition)

$$\| Tx_1 - Tx_2 \|_\infty = \max_{t\in\overline{I}} \| (Tx_1)(t) - (Tx_2)(t) \| \le \max_{t\in\overline{I}} \int_{t_0}^{t} \| f(s, x_1(s)) - f(s, x_2(s)) \| ds$$

$$\le \max_{t\in\overline{I}} L \int_{t_0}^{t} \| x_1(s) - x_2(s) \| ds \le L\overline{h} \max_{t\in\overline{I}} \| \lim_{x\to\infty} x_1(t) - x_2(t) \| = L\overline{h}\, \| x_1 - x_2 \|_\infty .$$

Since $0 < \alpha = L\overline{h} < 1$, so T is contractive. By Banach fixed point theorem, there exists a unique fixed point $x^* \in D$. That is,

$$x^*(t) = x_0 + \int_{t_0}^{t} f(s, x^*(s))ds,\ t \in \overline{I}.$$

This shows that (E) has a unique solution $x^*(t)$ on $\overline{I}$. □

Remark 3.6 It is seen that why h is replaced by $\overline{h}$, which is smaller, because the contraction of T is needed. This is acceptable because Picard theorem is a local result. However, we may employ a different norm for the Banach space $C(I)$ given by

$$\| x \|_* = \max_{t\in I} \| x(t) e^{-\lambda(t-t_0)} \|, \quad \forall x \in C(I),$$

where $\lambda > L$ and $I = [t_0 - h, t_0 + h]$ to show Picard theorem by Banach fixed point theorem for the following restatement. **(Left for Homework)**

Picard Theorem. Suppose that $f : Q \to R^n$

- is continuous;
- satisfies a Lipschitz condition.

Then the IVP (E) has a unique solution $x(t)$ at least defined on $I = [t_0 - h, t_0 + h]$, where $h = \min\{a, \frac{b}{M}\}$, $M = \max_{(t,x)\in Q} \| f(t,x) \|$.

Remark 3.7 The traditional proof of the above Picard theorem is given by Appendix A for your convenience.

Remark 3.8 If no Lipschitz condition holds, only existence can be assured by Peano theorem. However, its proof is quite different from Picard theorem.

3. Peano Theorem

1) Statement of Peano Theorem (Existence)

Peano Theorem Suppose that $f : Q \to R^n$ is continuous. Then the IVP (E) has a solution $x(t)$ on $I = [t_0 - h, t_0 + h]$, where $h = \min\{a, \frac{b}{M}\}$, $M = \max_{(t,x)\in Q} \| f(t,x) \|$.

2) Proof (Using Schauder Fixed Point Theorem)

Step 1. Define a Banach space $C(I)$, where $I = [t_0 - h, t_0 + h]$ with a norm given by $\| x \|_\infty = \max_{t\in I} \| x(t) \|$, $\forall x \in C(I)$.

Step 2. Define a closed subset of $C(I)$ as follows

$$D = \{x : x \in C(I),\ \| x - x_0 \|_\infty \le b,\ t \in I\} \subset C(I).$$

Show D is convex. For any $x_1, x_2 \in D$, we have

$$\| \lambda(x_1 - x_0) + (1-\lambda)(x_2 - x_0) \|_\infty \le \lambda \| x_1 - x_0 \|_\infty + (1-\lambda) \| (x_2 - x_0) \|_\infty \le \lambda b + (1-\lambda) b = b,$$

$$\Rightarrow \quad \lambda(x_1 - x_0) + (1-\lambda)(x_2 - x_0) \in D.$$ i.e. D is convex.

Therefore, D is a convex closed subset of $C(I)$.

Step 3. Define a mapping $T: D \to D$ as follows.

$$(Tx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds, \quad \forall t \in I, \quad \forall x \in D.$$

It is the same to show $T(D) \subset D$ like the one of Picard theorem. To show T is completely continuous on D is to show that T is continuous and on $T(D)$ is relatively compact in D.

Step 4. Show that $T(D)$ is relatively compact in D.

Being a family of functions defined on I, if T is uniformly bounded and equicontinuous, $T(D)$ is relatively compact in D.

Since $T(D) \subset D$, i.e. $\| Tx - x_0 \|_\infty \le b \quad \Rightarrow \quad \| Tx \|_\infty \le \| x_0 \|_\infty + b$, so T is uniformly bounded.

$\forall \varepsilon > 0$, taking $\delta = \dfrac{\varepsilon}{M} > 0$, for any $x \in D$ and any $t_1, t_2 \in I$, if $|t_1 - t_2| < \delta$, we have

$$\| Tx(t_2) - Tx(t_1) \|_\infty \le \| \int_{t_1}^{t_2} f(s, x(s))ds \|_\infty \le M \, |t_2 - t_1| < \varepsilon,$$

T is equicontinuous. Applying Ascoli-Arzela lemma yields $T(D)$ has a convergent subsequences in D. This shows that $T(D)$ is relatively compact in D by definition. By the way, we show that T is also continuous because of equicontinuity. Therefore, T is completely continuous on D.

Sept 5. Conclusion of Existence.

Since all the conditions are satisfied, it is concluded that there exists a fixed point $x^* \in D$ s.t. $Tx^* = x^*$ by Schauder fixed point theorem. That is,

$$x^*(t) = x_0 + \int_{t_0}^{t} f(s, x^*(s))ds, \; t \in I.$$

This shows that (E) has a solution $x^*(t)$ on I. □

Remark 3.9 Based on Peano theorem, we can show Picard theorem in a simple and different way.

Since f is continuous, the existence is done by Peano theorem. For uniqueness, if there are two solutions $x_1(t)$ and $x_2(t)$ s.t.

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_1(s))ds,\ \ t \in I\ ;$$

$$x_2(t) = x_0 + \int_{t_0}^{t} f(s, x_2(s))ds,\ \ t \in I\ .$$

Subtracting them yields

$$\| x_1(t) - x_2(t) \| = \left| \int_{t_0}^{t} \| f(s, x_1(s)) - f(s, x_2(s)) \| ds \right|$$

$$\le M \left| \int_{t_0}^{t} \| x_1(s) - x_2(s) \| ds \right| .$$

Without loss of generality, we assume that $t \in I^+ = [t_0, t_0 + h]$. Then,

$$\| x_1(t) - x_2(t) \| \le M \int_{t_0}^{t} \| x_1(s) - x_2(s) \| ds\ .$$

Application of Gronwall's inequality results in $\| x_1(t) - x_2(t) \| \le 0$, which is not possible unless $x_1(t) \equiv x_2(t)$, $t \in I^+$. It is similar for $t \in I^- = [t_0 - h, t_0]$. The uniqueness is done.

Remark 3.10 There are several ways to show the uniqueness. You are encouraged to do that by yourselves

Remark 3.11 The traditional proof of Peano theorem is given by Appendix B for your convenience. However, its proof is interesting to numerical approximation.

4. Summary

1) **Continuity $\Rightarrow$ Existence (Peano Theorem).**
2) **Lipschitz Condition $\Rightarrow$ Uniqueness (Gronwall's Inequality).**
3) **Continuity $+$ Lipschitz Condition $\Rightarrow$ Existence and Uniqueness (Picard Theorem or Peano Theorem $+$ Gronwall's Inequality).**
4) **Picard theorem and Peano theorem are local result.**
5) **Homework: Chapter One Exercises: Problem 12 and 17.**

5. Appendices

Appendix A. The Traditional Proof of Picard Theorem

Step 1. Based on the continuity of $f(t, x)$ on Q, it shows that C^1-solution of (E) $\Leftrightarrow$ the continuous solution of the integral equation as follows

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds\ ,\ \ (t, x) \in Q\ .$$

Step 2. Construct an iteration sequence $\{x_n(t)\}$, $t \in I$, as follows.

$$x_0(t) = x_0, \quad t \in I \;;$$

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\,ds, \quad t \in I \;;$$

$$\cdots$$

$$x_{n+1}(t) = x_0 + \int_{t_0}^{t} f(s, x_n(s))ds, \quad t \in I \;;$$

$$\cdots$$

Step 3. Show that $\{x_n(t)\}$ is well-defined, and $x_n(t)$ is continuous on I.

Since

$$\| x_n(x) - x_0 \| \le \int_{t_0}^{t} \| f(s, x_{n-1}(s)) \| ds \le M \,|t - t_0| \le M h \le b, \quad t \in I,$$

then $(t, x_n(t)) \in Q$ for all $n \in N^+$. This shows that $\{x_n(t)\}$ is well-defined.

The continuity of $f(t, x)$ on Q implies that $x_n(t)$ is continuous on I.

Step 4. Show that $\{x_n(t)\}$ is uniformly convergent on I.

If there exists $k \ge 1$ s.t. $x_{k+1}(t) = x_k(t)$, $t \in I$; i.e.

$$x_k(t) = x_0 + \int_{t_0}^{t} f(s, x_k(s))ds, \quad t \in I,$$

then $x_k(t)$ is a C^1-solution of (E). The existence is shown. If for all $k \ge 1$ s.t. $x_{k+1}(t) \ne x_k(t)$, then we can get an infinite sequence $\{x_n(t)\}$ defined above, called a **Picard sequence**.

Since the uniformly convergence of $\{x_n(t)\}$ on I is equivalent to the uniformly convergence of the following series

$$x_0(t) + \sum_{n=0}^{\infty} [x_{n+1}(t) - x_n(t)] \quad \text{on } I,$$

we show that the series is uniformly convergent on I. To this end, we have

$$\| x_1(t) - x_0(t) \| \le M \,|t - t_0|;$$

$$\begin{aligned} \| x_2(t) - x_1(t) \| &= \left\| \int_{t_0}^{t} [f(s, x_1(s)) - f(s, x_0(s))] ds \right\| \\ &\le \left| \int_{t_0}^{t} \| f(s, x_1(s)) - f(s, x_0(s)) \| ds \right| \\ &\le L \left| \int_{t_0}^{t} \| x_1(s) - x_0(s) \| ds \right| \end{aligned}$$

$$\leq L\left|\int_{t_0}^{t}|s-t_0|\,ds\right|.$$

Since s is within t_0 and t, so $(s-t_0)(t-t_0)>0$, we have

$$\|x_2(t)-x_1(t)\|\leq ML\left|\int_{t_0}^{t}(s-x_0)ds\right|=\frac{ML}{2!}|t-t_0|^2,\quad t\in I.$$

Similarly, we have

$$\|x_3(x)-x_2(x)\|=\left|\int_{t_0}^{t}[f(s,x_2(s))-f(s,x_1(s))]ds\right|$$

$$\leq L\left|\int_{t_0}^{t}\|x_2(s)-x_1(s)\|\,ds\right|$$

$$\leq \frac{ML^2}{2!}\left|\int_{t_0}^{t}(s-t_0)^2ds\right|=\frac{ML^2}{3!}|t-t_0|^3,\quad t\in I.$$

Suppose that we have

$$\|x_n(t)-x_{n-1}(t)\|\leq\frac{ML^{n-1}}{n!}|t-t_0|^n,\quad t\in I;$$

then,

$$\|x_{n+1}(t)-x_n(t)\|=\left\|\int_{t_0}^{t}[f(s,x_n(s))-f(s,x_{n-1}(s))]ds\right\|$$

$$\leq\left|\int_{t_0}^{t}\|f(s,x_n(s))-f(s,x_{n-1}(s))\|\,ds\right|$$

$$\leq\frac{ML^n}{n!}\left|\int_{t_0}^{t}|s-t_0|^n\,ds\right|$$

$$\leq\frac{ML^n}{(n+1)!}|t-t_0|^{n+1},\quad t\in I.$$

This shows that for any $n\in N^+$, we have

$$\|x_{n+1}(t)-x_n(t)\|\leq\frac{ML^n}{(n+1)!}|t-t_0|^{n+1},\quad t\in I.$$

Since the series $\sum_{n=0}^{\infty}\frac{ML^n}{(n+1)!}|t-t_0|^{n+1}$ is uniformly convergent on $t\in I$, so does

$x_0(t)+\sum_{n=0}^{\infty}[x_{n+1}(t)-x_n(t)]$ on I.

Step 5. Show that the following integral equation

$$x(t)=x_0+\int_{t_0}^{t} f(s,x(s))ds\,,\quad (t,x)\in Q\,,$$

has a continuous solution.

Let $\lim_{n\to\infty} x_n(t)=x^*(t)$, $t\in I$. Since $x_n(t)$ is continuous on I, so is $x^*(t)$. On the other way,

$$\| f(t,x_n(t))-f(t,x^*(t))\|\le L\| x_n(t)-x^*(t)\|.$$

Therefore, $f(t,x_n(t))$ is uniformly convergent to $f(t,x^*(t))$ on I. Then,

$$\lim_{n\to\infty}\int_{t_0}^{t} f(s,x_n(s))ds=\int_{t_0}^{t} f(s,x^*(s))ds\,.$$

Taking limit on both sides of

$$x_{n+1}(t)=x_0+\int_{t_0}^{t} f(s,x_n(s))ds,\quad t\in I\,,$$

yields

$$x^*(t)=x_0+\int_{t_0}^{t} f(s,x^*(s))ds,\quad t\in I\,.$$

Step 6. The uniqueness is the same to Remark 9. □

Appendix B. The Traditional Proof of Peano Theorem

1) Peano Theorem If $f\in C(Q)$, then, there exists $x(t)$ of (E) defined on $I=[t_0-h,t_0+h]$, where $h=\min\{a,\frac{b}{M}\}$, $M\ge \sup_{(t,x)\in Q}\| f(t,x)\|$.

2) Detailed Proof

Construction for $t\in I_+$ only (the construction for $t\in I_-$ is similar).

Step 1. Construction:

For each $m\ge 1$, we subdivide I_+ with $I_+=\bigcup_{k=1}^{m}[t_{k-1}^{(m)},\, t_k^{(m)}]$, where $t_k^{(m)}=t_0+\frac{hk}{m}$ for $k=1,2,\cdots,m$. We define $x_m(t)$ step by step on each subinterval $[t_{k-1}^{(m)},\, t_k^{(m)}]$:

$$x_m(t)=x_0+f(t_0,x_0)(t-t_0)\quad\text{for}\quad t\in[t_0^{(m)},t_1^{(m)}],\ \text{where}\quad t_0^{(m)}=t_0\,,$$

$$\Rightarrow\quad x_m(t_1^{(m)})=x_0+f(t_0,x_0)(t_1^{(m)}-t_0)\,;$$

$$x_m(t)=x_m(t_1^{(m)})+f(t_1^{(m)},x_m(t_1^{(m)}))(t-t_1^{(m)})\quad\text{for}\quad t\in[t_1^{(m)},\, t_2^{(m)}]\,,$$

$$\Rightarrow\quad x_m(t_2^{(m)})=x_m(t_1^{(m)})+f(t_1^{(m)},x_m(t_1^{(m)}))(t_2^{(m)}-t_1^{(m)})\,;$$

By induction, if we have constructed $x_m(t) = x_m(t_{k-1}^{(m)}) + f(t_{k-1}^{(m)}, x_m(t_{k-1}^{(m)}))(t - t_{k-1}^{(m)})$ for $t \in [t_{k-1}^{(m)}, t_k^{(m)}]$, so we have $x_m(t_k^{(m)}) = x_m(t_{k-1}^{(m)}) + f(t_{k-1}^{(m)}, x_m(t_{k-1}^{(m)}))(t_k^{(m)} - t_{k-1}^{(m)})$. Then, we define

$$x_m(t) = x_m(t_k^{(m)}) + f(t_k^{(m)}, x_m(t_k^{(m)}))(t - t_k^{(m)}) \quad \text{for} \quad t \in [t_k^{(m)}, t_{k+1}^{(m)}].$$

So we have defined $x_m(t)$ on all $t \in I_+$, which is called the Euler polygons.

Step 2. $\{x_m(t)\}$ is well defined on I_+:

Since

$$\begin{aligned}
\| x_m(t) - x_0 \| &\le \| x_m(t) - x_m(t_k^{(m)}) \| + \| x_m(t_k^{(m)}) - x_m(t_{k-1}^{(m)}) \| + \cdots + \| x_m(t_1^{(m)}) - x_0 \| \\
&= \| f(t_k^{(m)}, x_m(t_k^{(m)}))(t - t_k^{(m)}) \| + \| f(t_{k-1}^{(m)}, x_m(t_{k-1}^{(m)}))(t_{k-1}^{(m)} - t_k^{(m)}) \| \\
&\quad + \cdots + \| f(t_0, x_0)(t_1^{(m)} - t_0) \| \\
&\le M(t - t_k^{(m)}) + M(t_{k-1}^{(m)} - t_k^{(m)}) + \cdots + M(t_1^{(m)} - t_0) \\
&= M(t - t_0) \le Mh \le b, \quad t_0 \le t \le t_{k+1}^{(m)}.
\end{aligned}$$

So that $(t, x_m(t)) \in Q$ for $t_0 \le t \le t_{k+1}^{(m)}$ $\Rightarrow$ $\{x_m(t)\}$ is well defined on I_+.

Step 3. Since $x_m(t)$ is continuous at $t_k^{(m)}$, and $x_m(t)$ has a derivative $f(t_k^{(m)}, x_m(t_k^{(m)}))$ on $[t_k^{(m)}, t_{k+1}^{(m)}]$, then,

$$x_m(t) = x_0 + \int_{t_0}^{t} f^{(m)}(s)ds \quad \text{for} \quad t_0 \le t \le t_{k+1}^{(m)},$$

where $f^{(m)}(t) := f(t_j^{(m)}, x_m(t_j^{(m)}))$ for $t \in [t_j^{(m)}, t_{j+1}^{(m)}]$ (piecewise function).

Step 4. Show that $\{x_m(t)\}$ is equicontinuous and uniformly bounded.

For $t', t \in I_+$,

$$\| x_m(t) - x_m(t') \| = \| \int_t^{t'} f^{(m)}(s)ds \| \le M\,|t - t'|, \quad \forall\, m \ge 1$$

and

$$\| x_m(t) \| \le \| x_m(t_0) \| + \| x_m(t) - x_m(t_0) \| \le \| x_m(t_0) \| + Mh,$$

hence, $\{x_m(t)\}$ is equicontinuous and uniformly bounded.

Step 5. Applying Ascoli–Arzela lemma, we have $x_{m_j}(t) \overset{E}{\to} x(t)$ on I_+, where $\{x_{m_j}(t)\} \subseteq \{x_m(t)\}$. We claim that $x(t)$ is the desired solution of (E) if $f^{(m_j)}(t) \overset{E}{\to} f(t, x(t))$ on I_+.

Step 6. Show that $f^{(m_j)}(t) \overset{E}{\to} f(t, x(t))$ on I_+:

For simple notation, we suppose that $x_m(t) \overset{E}{\to} x(t)$ on I_+. Then, for the given $\varepsilon > 0$, since $f \in C(Q)$ and Q is compact, $\exists\, \delta > 0$ such that

$$\| f(t, x) - f(t', x') \| < \varepsilon$$

whenever $\| (t - t', x - x')^T \| < \delta$, (uniformly continuous).

Now choose m so large such that $\dfrac{h}{m} < \dfrac{\delta}{3}$, $\dfrac{Mh}{m} < \dfrac{\delta}{3}$ and $\| x_m(t) - x(t) \| < \dfrac{\delta}{3}$ whenever $t \in I_+$. For $t \in I_+$, $t \in [t_k^{(m)}, t_{k+1}^{(m)}]$ for some k, we have

$$\begin{aligned}\| (t_k^{(m)} - t, x_m(t_k^{(m)}) - x(t))^T \| &\le | t_k^{(m)} - t | + \| x_m(t_k^{(m)}) - x(t)) \| \\ &\le | t_k^{(m)} - t | + \| x_m(t_k^{(m)}) - x_m(t) \| + \| x_m(t) - x(t) \| \\ &\le \frac{h}{m} + \frac{Mh}{m} + \frac{\delta}{3} \le \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta,\end{aligned}$$

and hence

$$\| f^{(m)}(t) - f(t, x(t)) \| = \| f^{(m)}(t_k^{(m)}, x_m(t_k^{(m)})) - f(t, x(t)) \| < \varepsilon. \quad \square$$